Identities involving the bihyperbolic functions

Introduction

Biexponential component functions

The asymmetric biexponential component functions were defined previously [1] as follows.

$$(u,v)^{(n,m)} \stackrel{\text{def}}{=} \frac{u^n}{n!} \cdot \frac{v^m}{m!}$$

The asymmetric component functions satisfy the following identity.

$$(u,v)^{(m,n)} = (v,u)^{(n,m)}$$

If one defines

$$\frac{1}{(-n)!} \stackrel{\text{def}}{=} 0$$

for every positive integer *n*, then $(u,v)^{(m,n)} = 0$ whenever *m* or *n* is negative.

Biexponential function

The biexponential function was defined previously as an infinite sum of the symmetric component functions. However, it can be defined equivalently a an infinite sum of the asymmetric component functions as follows.

$$e^{(u;v)} \stackrel{\text{def}}{=} e^{(u,v)} \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} (u,v)^{(n,n)} = (u,v)^{(0,0)} + (u,v)^{(1,1)} + (u,v)^{(2,2)} + (u,v)^{(3,3)} + \cdots$$

Biexponential derivative functions

The 0^{th} biexponential derivative function is, by definition, identical to the biexponential function

$$0^{th}(u,v) \stackrel{\text{def}}{=} e^{(u,v)} \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} (u,v)^{(n,n)} = (u,v)^{(0,0)} + (u,v)^{(1,1)} + (u,v)^{(2,2)} + (u,v)^{(3,3)} + \cdots$$

and (as before) the positive biexponential derivative functions are defined for every positive integer *n* by

$$n^{th}(u,v) \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty} (u,v)^{(i,i+n)} = (u,v)^{(0,n)} + (u,v)^{(1,n+1)} + (u,v)^{(2,n+2)} + (u,v)^{(3,n+3)} + \ \dots .$$

The negative biexponential derivative functions are defined similarly.

$$(-n)^{th}(u,v) \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty} (u,v)^{(i+n,i)} = (u,v)^{(n,0)} + (u,v)^{(n+1,1)} + (u,v)^{(n+2,2)} + (u,v)^{(n+3,3)} + \ \dots.$$

If n and m are natural numbers, then we have

$$(n-m)^{th}(u,v) = \sum_{i=-\infty}^{\infty} (u,v)^{(i+m,\ \ i+n)} .$$

Identities involving the biexponential derivative functions

Employing the definition of the biexponential derivative functions we have the following identities. For every integer *n*,

$$n^{th}(u,v) = \ (-n)^{th}(v,u),$$

$$\frac{\partial}{\partial u} n^{th}(u,v) = (n+1)^{th}(u,v),$$

$$\frac{\partial}{\partial v} n^{th}(u,v) = (n-1)^{th}(u,v)$$

and, in particular,

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v} n^{th}(u,v) = n^{th}(u,v).$$

The biexponential derivative proposition

I previously proposed (but did not prove) that the biexponential derivative functions are related to the exponential function by the following identity.

$$e^{u+v} = \sum_{n=-\infty}^{\infty} n^{th}(u,v)$$

The bihyperbolic functions and their derivatives

Given the above relationship between the exponential function and the biexponential derivative functions, we expect generalizations of the hyperbolic functions to exist that have a similar relationship to the single parameter hyperbolic functions.

Method

We define the bihyperbolic functions analogously to the biexponential function. For every pair of natural numbers n and m we define

$$\cosh^{(n-m)}(u,v) \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty} (u,v)^{(2i+m,\ 2i+n)}.$$

Results

We observe that the basic identities for the bihyperbolic functions are similar to the basic identities for the biexponential derivative functions.

$\cosh^{(n-m)}(u,v) \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty}(u,v)^{(2i+m,\ 2i+n)} = \sum_{i=-\infty}^{\infty}(v,u)^{(2i+n,\ 2i+m)} = \cosh^{(m-n)}(v,u),$

$\frac{\partial}{\partial u}\cosh^{(n-m)}(u,v) = \sum_{i=-\infty}^{\infty}\frac{\partial}{\partial u}(u,v)^{(2i+m,\ 2i+n)} = \sum_{i=-\infty}^{\infty}(u,v)^{(2i+m,\ 2i+n+1)} = \cosh^{(n+1-m)}(u,v),$

$\frac{\partial}{\partial v}\cosh^{(n-m)}(u,v) = \sum_{i=-\infty}^{\infty}\frac{\partial}{\partial v}(u,v)^{(2i+m,\ 2i+n)} = \sum_{i=-\infty}^{\infty}(u,v)^{(2i+m+1,\ 2i+n)} = \cosh^{(n-m-1)}(u,v)$

and

$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\cosh^{(n-m)}(u,v) = \cosh^{(n+1-m-1)}(u,v) = \cosh^{(n-m)}(u,v).$

Discussion

I previously suggested that the exponential function may be related to the biexponential derivatives by the following identity.

$$e^{u+v} = \sum_{n=-\infty}^{\infty} n^{th}(u, v)$$

If this identity holds, then one my expect the bihyperbolic functions to be related to the hyperbolic functions according to the following identities.

$$\cosh(u + v) = \sum_{n=-\infty}^{\infty} \cosh^{(2n)}(u, v)$$

and

$$\sinh(u + v) = \sum_{n=-\infty}^{\infty} \cosh^{(2n+1)}(u, v).$$

tags

biexponential; bihyperbolic; derivative; component; function; asymmetric;