Derivatives of the biexponential function

tags

biexponential function; derivatives

Introduction

As discussed previously [1], the biexponential function $e^{(u;v)}$ resembles the exponential function in that it has the property that its double derivative with respect to the $\frac{\partial}{\partial u}\frac{\partial}{\partial v}$ operator is equal to itself.

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}e^{(u;v)} = e^{(u;v)}$$

The biexponential function is defined as follows.

$$e^{(u;v)} \stackrel{\text{def}}{=} (u;v)^0 + (u;v)^1 + (u;v)^2 + (u;v)^3 + \cdots$$

where

$$(u;v)^n \stackrel{\text{def}}{=} (u,v)^{(n,n)}$$

and where

$$(u,v)^{(n,m)} \stackrel{\text{def}}{=} \frac{u^n}{n!}\cdot\frac{v^m}{m!}.$$

Theorem

$$e^{(u;v)} = e^{(v;u)}$$

proof

$$e^{(u;v)} = (u;v)^0 + (u;v)^1 + (u;v)^2 + (u;v)^3 + \cdots$$

$$= (u,v)^{(0,0)} + (u,v)^{(1,1)} + (u,v)^{(2,2)} + (u,v)^{(3,3)} + \cdots$$

$$= \frac{u^0}{0!}\cdot\frac{v^0}{0!} + \frac{1^n}{1!}\cdot\frac{1^m}{1!} + \frac{2^n}{2!}\cdot\frac{2^m}{2!} + \frac{3^n}{3!}\cdot\frac{3^m}{3!} + \cdots$$

$$= \frac{v^0}{0!}\cdot\frac{u^0}{0!} + \frac{1^m}{1!}\cdot\frac{1^n}{1!} + \frac{2^m}{2!}\cdot\frac{2^n}{2!} + \frac{3^m}{3!}\cdot\frac{3^n}{3!} + \cdots$$

$$= (v,u)^{(0,0)} + (v,u)^{(1,1)} + (v,u)^{(2,2)} + (v,u)^{(3,3)} + \cdots$$

$$= (v;u)^0 + (v;\ u)^1 + (v;\ u)^2 + (v;\ u)^3 + \cdots$$

$$= e^{(u;v)}$$

Theorem (the derivative theorem)

Let k ≥ l be natural numbers, let n = k – l, and let $n^{th}(u, v)$ be the function defined as follows,

$$n^{th}(u,v) \stackrel{\text{def}}{=} (u,v)^{(0,n)} + (u,v)^{(1,n+1)} + (u,v)^{(2,n+2)} + (u,v)^{(3,n+3)} + (u,v)^{(4,n+4)} + \ \dots$$

then

$$\frac{\partial^k}{\partial u^k}\frac{\partial^l}{\partial v^l}e^{(u;v)} = n^{th}(u,v)$$

and

$$\frac{\partial^l}{\partial u^l}\frac{\partial^k}{\partial v^k}e^{(u;v)} = n^{th}(v,u).$$

In particular,

$$e^{(u;v)} = 0^{th}(u,v).$$

proof

$$\frac{\partial^k}{\partial u^k}\frac{\partial^l}{\partial v^l}e^{(u;v)} = \frac{\partial^n}{\partial u^n}e^{(u;v)} \text{ (assuming continuity)}$$

$$= \frac{\partial^n}{\partial u^n}[(u;v)^0 + (u;v)^1 + (u;v)^2 + (u;v)^3 + \cdots]$$

$$= \frac{\partial^n}{\partial u^n}\left[(u,v)^{(0,0)} + (u,v)^{(1,1)} + (u;v)^{(2,2)} + (u;v)^{(3,3)} + \cdots\right]$$

$$= \frac{\partial^n}{\partial u^n}(u,v)^{(0,0)} + \frac{\partial^n}{\partial u^n}(u,v)^{(1,1)} + \frac{\partial^n}{\partial u^n}(u,v)^{(2,2)} + \frac{\partial^n}{\partial u^n}(u,v)^{(3,3)} + \cdots$$

$$= \frac{\partial^n}{\partial u^n}\frac{u^0}{0!}\cdot\frac{v^0}{0!} + \frac{\partial^n}{\partial u^n}\frac{u^1}{1!}\cdot\frac{v^1}{1!} + \frac{\partial^n}{\partial u^n}\frac{u^2}{2!}\cdot\frac{v^2}{2!} + \frac{\partial^n}{\partial u^n}\frac{u^3}{3!}\cdot\frac{v^3}{3!} + \cdots$$

$$= \sum_{i=0}^{n-1} 0 + \sum_{i=n}^{\infty}\frac{u^{i-n}}{(i-n)!}\cdot\frac{v^i}{i!} = \sum_{i=n}^{\infty}\frac{u^{i-n}}{(i-n)!}\cdot\frac{v^i}{i!} = \sum_{i=n}^{\infty}(u,v)^{(i-n,n)}$$

$$= (u,v)^{(0,n)} + (u,v)^{(1,n+1)} + (u,v)^{(2,n+2)} + (u,v)^{(3,n+3)} + (u,v)^{(4,n+4)} + \ \dots$$

$$= n^{th}(u,v)$$

Also,

$$\frac{\partial^l}{\partial u^l}\frac{\partial^k}{\partial v^k}e^{(u;v)} = \frac{\partial^n}{\partial v^n}e^{(u;v)} \text{ (assuming continuity)}$$

$$= \frac{\partial^n}{\partial v^n}e^{(v;u)}$$

$$= n^{th}(v,u)$$