The fundamental theorem of the biexponential function

tags

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Introduction

The exponential function is a single variable function which is equal to its own derivative. The question arises: are their multivariable functions which also equal to their own derivatives? The answer to this question turns out to be yes, in the following sense. There exists a two parameter function which when differentiated with respect to both parameters yields itself. I will refer to this function as the biexponential function.

Definition

$$e^{(u;v)} \stackrel{\text{def}}{=} (u;v)^0 + (u;v)^1 + (u;v)^2 + (u;v)^3 + \cdots$$

where

$$(u;v)^n \stackrel{\text{def}}{=} (u,v)^{(n,n)}$$

and where

$$(u,v)^{(n,m)} \stackrel{\text{def}}{=} \frac{u^n}{n!} \cdot \frac{v^m}{m!}$$

First Theorem of the biexponential function

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v} e^{(u;v)} = e^{(u;v)}$$

proof

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v} e^{(u;v)} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}\left[(u;v)^0 + (u;v)^1 + (u;v)^2 + (u;v)^3 + \cdots\right]$$

$$= \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u;v)^0 + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u;v)^1 + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u;v)^2 + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u;v)^3 + \cdots$$

$$= \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u,v)^{(0,0)} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u,v)^{(1,1)} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u,v)^{(2,2)} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}(u,v)^{(3,3)} + \cdots$$

$$= \frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{u^0}{0!}\cdot\frac{v^0}{0!} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{u^1}{1!}\cdot\frac{v^1}{1!} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{u^2}{2!}\cdot\frac{v^2}{2!} + \frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{u^3}{3!}\cdot\frac{v^3}{3!} + \cdots$$

$$= 0 + \frac{u^0}{0!}\cdot\frac{v^0}{0!} + \frac{u^1}{1!}\cdot\frac{v^1}{1!} + \frac{u^2}{2!}\cdot\frac{v^2}{2!} + \cdots$$

$$= e^{(u;v)}$$